ScienceDirect

IFAC Papers Online CONFERENCE PAPER ARCHIVE

IFAC PapersOnLine 50-1 (2017) 1334–1339

Robustness issues in double-integrator undirected rigid formation systems

Zhiyong Sun * **Brian D. O. Anderson** *,** **Shaoshuai Mou** *** **A. Stephen Morse** ****

* *Data61 - CSIRO and Research School of Engineering, Australian National University, Canberra ACT 2601, Australia.*
** *School of Automation, Hangzhou Dianzi University, Hangzhou 310018, China.*
*** *School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN, USA.*
**** *Department of Electrical Engineering, Yale University, New Haven, CT, USA.*

Abstract: In this paper we consider rigid formation control systems modelled by double integrators (including formation stabilization systems and flocking control systems), with a focus on their robustness property in the presence of distance mismatch. By introducing additional state variables we show the augmented double-integrator distance error system is self-contained, and we prove the exponential stability of the distance error systems via linearization analysis. As a consequence of the exponential stability, the distance error still converges in the presence of small and constant distance mismatches, while additional motions of the resulted formation will occur. We further analyze the rigid motions induced by constant mismatches for both double-integrator formation stabilisation systems and flocking control systems.

Keywords: Formation control; exponential stability, double-integrator systems; distance mismatch; rigid body motion.

1. INTRODUCTION

Formation control for a group of autonomous mobile agents has gained much attention due to its broad applications in many areas including both civil and military fields Oh et al. (2015). In this paper we focus on formation control strategy based on graph rigidity theory, motivated by its many advantages over other formation control strategies, such as its independence of a global coordinate system Oh et al. (2015); Olfati-Saber and Murray (2002); Krick et al. (2009). The rigidity-based formation control has received much attention in recent years, in particular since the comprehensive analysis on the stability and convergence conducted in Krick et al. (2009).

One of the main concerns when implementing any formation controller in practice is the robustness issue in the presence of distance measurement error, perturbations or information inconsistency in distributed coordination. It has been shown in Belabbas et al. (2012) by using a 2-D rigid triangular formation as an example that undirected formations may display undesired motions induced by *distance mismatch* (a term that describes inconsistency in either distance measurements or desired distance specification for a particular edge of the formation, as viewed from the two agents on which it is incident). A more comprehensive study for general 2-D rigid undirected formations is reported in Mou et al. (2016), which shows that for any 2-D rigid formations, circular motion will almost surely occur as a consequence of constant distance mismatch. A corresponding study for 3-D rigid formations with mismatched distances is reported in Sun et al. (2017) which proves that generically a helical motion in 3-D rigid formations could be induced by constant distance mismatch. Recent efforts on how to eliminate undesired motions induced by mismatch distances or how to generate rigid motions by considering distance mismatches as control parameters are also available, see e.g. Mou et al. (2014); Garcia de Marina et al. (2015, 2016),

We note that most results on rigid formation control reported in the literature (including the above mentioned papers) are based on simple single-integrator formation models. Such models allow one to focus on the stability and convergence of the formation dynamics, while the kinematics for each agent have been ignored. As a comparison, a double-integrator agent model is considered to be a more suitable model to describe real-life formation control tasks as the control input relates to the acceleration instead of velocity, as in single-integrator formation models. Double-integrator models have also been very popular in studying distributed coordination among spatially distributed agents, such as flocking control of multi-

⋆ This work was supported by the Australian Research Council under grant DP130103610 and DP160104500. Z. Sun was supported by the Australian Prime Minister's Endeavour Postgraduate Award from Australian Government. The work of S. Mou was supported by a funding from Northrop Grumman Corporation.
The authors also thank Héctor Garcia de Marina for helpful discussions. E-mails: {zhiyong.sun, brian.anderson}@anu.edu.au; mous@purdue.edu; as.morse@yale.edu.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2017.08.221

agent systems Olfati-Saber (2006),Tanner et al. (2007). In recent years, rigid formation control modelled by double-integrator agents has also begun to attract much attention; see e.g. Deghat et al. (2016) on flocking control of rigid formation by combining distance-based shape control and velocity consensus, and Sun et al. (2016a) which focused on the system dynamics and stability analysis of different equilibria for double-integrator rigid formation control systems (including the shape stabilization system and flocking system). However, a robustness analysis on double-integrator formation systems is still lacking (with the exception of Garcia de Marina et al. (2017)). Note that the paper Garcia de Marina et al. (2017) discussed the robustness issue in rigid shape stabilization control for second-order agents with distance mismatch, but did not consider the inclusion of an additional flocking requirement. Also, in contrast to the stability analysis of Garcia de Marina et al. (2017), we will emphasize the important issue of augmenting a self-contained formation error system for the stability and perturbation analysis of double-integrator formation systems.

Following the spirit of the above mentioned papers (especially Mou et al. (2016) and Deghat et al. (2016)), we aim to provide a comprehensive analysis on robustness issues in double-integrator formation systems with mismatched distances. The aims and contributions of this paper are

- to revisit the stability results of formation systems governed by a double-integrator version of the standard single-integrator control law in Krick et al. (2009). Two types of double-integrator formation systems will be considered in this paper, namely, the formation stabilization system and formation flocking system (the definitions will be made clear in Section 2);
- to derive self-contained equations for the evolution of distance error systems (the definition of such systems will be made clear in Section 3). Compared to the case of single-integrator system models discussed in Mou et al. (2016), for the self-contained issue of double-integrator formation systems, the angular momentum get involved too;
- to establish the exponential stability of the linearized distance error systems, which is crucial for the study of the robustness property against distance perturbations (i.e. small distance mismatches considered in this paper);
- to determine the rigid body motion properties for double-integrator formation systems for shape control and double-integrator flocking systems in the presence of small and constant mismatched distances.

The paper is organized as follows. In Section 2, preliminary concepts on graph theory, rigidity theory are introduced. We also review in Section 2 two types of formation system equations and a known convergence result. In Section 3, we discuss the self-contained distance error systems by augmenting additional state variables and further show its local exponential stability at the origin via linearization analysis. Section 4 focuses on the robustness issues of the double-integrator formation systems and flocking systems. Finally, Section 5 concludes this paper.

2. PRELIMINARIES

2.1 Graph rigidity and notations

Consider an undirected graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. We define an *orientated* incidence matrix $H \in \mathbb{R}^{m\times n}$ for the *undirected* graph $\mathcal{G}$ by assigning an *arbitrary* orientation for each edge. Note that for a rigid formation modelled by an *undirected* graph considered in this paper, the orientation of each edge for writing the incidence matrix can be defined arbitrarily and the stability analysis in the next sections remains unchanged. Following this, we define the entries of H as $h_{ki} = +1$ if the k-th edge sinks at node i, or $h_{ki} = -1$ if the k-th edge leaves node i, or $h_{ki} = 0$ otherwise. The Laplacian matrix $L(\mathcal{G})$ is also often used for matrix representation of a graph $\mathcal{G}$, which is defined as $L(\mathcal{G}) = H^\top H$ for undirected graphs. For a connected undirected graph, there holds $\text{rank}(L) = n - 1$ and $\text{null}(L) = \text{null}(H) = \text{span}\{\mathbf{1}_n\}$.

We denote by $p = [p_1^\top, p_2^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{dn}$ the stacked vector of all the agents' positions $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^d$. The incidence matrix H defines the sensing topology of the formation, i.e. it encodes the set of available relative positions that can be measured by the agents. By introducing the matrix $\bar{H} := H \otimes I_d$, one can construct the stacked vector z of available relative positions by

$$z = \bar{H}p, \tag{1}$$

where each element $z_k \in \mathbb{R}^d$ in z is the relative position vector for the vertex pair defined by the edge $\mathcal{E}_k$.

This paper focuses on formation control of rigid shapes. The definition of graph rigidity can be found in e.g. Hendrickson (1992). Define $Z(z) = \text{diag}(z_1, z_2, \ldots, z_m) \in \mathbb{R}^{dm\times m}$. With this notation at hand, we consider the smooth distance map $r_\mathcal{G} : \mathbb{R}^{dn} \longrightarrow \mathbb{R}^m, r_\mathcal{G}(p) = (\|p_i - p_j\|^2)_{(i,j)\in\mathcal{E}} = Z^\top z$. A useful tool to study graph rigidity is the **rigidity matrix**, which is defined as the Jacobian matrix $R(p) = \frac{1}{2}\partial r_\mathcal{G}(p)/\partial(p) = Z(z)^\top \bar{H} \in \mathbb{R}^{m\times dn}$. A framework $(\mathcal{G}, p)$ is *infinitesimally rigid* if $\text{rank}(R(z)) = 2n-3$ when it is embedded in $\mathbb{R}^2$ or if $\text{rank}(R(z)) = 3n-6$ when it is embedded in $\mathbb{R}^3$. Additionally, if $|\mathcal{E}| = 2n - 3$ in the 2-D case or $|\mathcal{E}| = 3n - 6$ in the 3-D case then the framework is called *minimally rigid*.

2.2 System equations

Let $d_{k_{ij}}$ denote the desired length of edge k which links agents i and j. We assume that the set of desired lengths is realizable, i.e., there exists a formation in $\mathbb{R}^d$ whose inter-agent distances correspond to the desired values. In the following, the set of all formations $(\mathcal{G}, p)$ which satisfies the distance constraints is referred to as the set of *target formations*. In this paper we assume that all target formations are *infinitesimally and minimally rigid*. We further define (for an arbitrary formation) $e_{k_{ij}} = \|p_i - p_j\|^2 - d_{k_{ij}}^2 = \|z_k\|^2 - d_{k_{ij}}^2$ to denote the squared distance error for edge k. Note we may also use e_k and d_k occasionally for notational convenience in the sequel if no confusion is expected. The distance error vector is denoted by $e = [e_1, e_2, \ldots, e_m]^\top$.

Define

$$\psi(p, v) := \frac{1}{2}\sum_{i\in\mathcal{V}} \|v_i\|^2 + V(p), \tag{2}$$

where v_i is the velocity of agent i (i.e. $v_i := \dot{p}_i$), and $V(p) = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}}(\|p_i - p_j\|^2 - d^2_{k_{ij}})^2$ is the commonly-used potential for shape stabilization (see e.g. Krick et al. (2009)).

(i) Formation stabilization system The formation stabilization system (without velocity consensus term) modelled by double integrators is described by the following equations:

$$\begin{aligned}
&\dot{p}_i = v_i,\\
&\dot{v}_i = -\alpha v_i - \sum_{j\in\mathcal{N}_i}\left(\|p_i - p_j\|^2 - d^2_{k_{ij}}\right)(p_i - p_j),
\end{aligned} \tag{3}$$

where α is a positive velocity damping parameter. In a compact form, the above system equation can be rewritten as

$$\begin{aligned}
\dot{p} &= \nabla_v \psi = v,\\
\dot{v} &= -\alpha\nabla_v\psi - \nabla_p\psi = -\alpha v - R^\top(z)e(z),
\end{aligned} \tag{4}$$

where R is the rigidity matrix for the formation.

For the stability analysis it will be more convenient to focus on the relative position dynamics (i.e. the system $\dot{z}_i$). From (1) and (4) one can derive the relative position dynamics as follows

$$\begin{aligned}
\dot{z} =& \bar{H}\dot{p} = \bar{H}v,\\
\ddot{z} =& \bar{H}\dot{v} = -\alpha\bar{H}v - \bar{H}R^\top(z)e(z)\\
=& -\alpha\dot{z} - \bar{H}R^\top(z)e(z).
\end{aligned} \tag{5}$$

(ii) Formation flocking system The formation flocking system (with a velocity consensus term in an undirected interaction graph) can be written as

$$\begin{aligned}
&\dot{p}_i = v_i,\\
&\dot{v}_i = \alpha\sum_{j\in\mathcal{N}_i}(v_j - v_i) - \sum_{j\in\mathcal{N}_i}\left(\|p_i - p_j\|^2 - d^2_{k_{ij}}\right)(p_i - p_j),
\end{aligned} \tag{6}$$

where α is a positive gain for velocity consensus. The above system equation can be rewritten in a compact form

$$\begin{aligned}
\dot{p} &= \nabla_v\psi = v,\\
\dot{v} &= -\alpha\bar{L}v - R^\top(z)e(z),
\end{aligned} \tag{7}$$

where $\bar{L} = L \otimes I_d$ and L is the Laplacian matrix for the underlying undirected and connected graph.

By using the same approach of obtaining the relative position dynamics in (5), one can also obtain the relative position dynamics $\dot{z}$ for the formation flocking system (7) as follows

$$\begin{aligned}
\dot{z} =& \bar{H}\dot{p} = \bar{H}v,\\
\ddot{z} =& \bar{H}\dot{v} = -\alpha\bar{H}\bar{L}v - \bar{H}R^\top(z)e(z)\\
=& -\alpha\bar{H}\bar{H}^\top\dot{z} - \bar{H}R^\top(z)e(z).
\end{aligned} \tag{8}$$

It is easy to see that this differs from the equations (5) for formation shape stabilization in respect of only the term involving $\dot{z}$.

2.3 A known result on asymptotic convergence

The following convergence result has been well established in the literature under different contexts; see e.g. Dimarogonas and Johansson (2008); Deghat et al. (2016); Sun et al. (2016a).

Lemma 1. The formation stabilization system (4) (without velocity consensus term) is locally asymptotically stable in that $e \to 0$ and all agents' velocities converge to zero asymptotically. Furthermore, the formation flocking system (7) (with velocity consensus term) is locally asymptotically stable in that $e \to 0$ and all the velocities reach consensus asymptotically.

The proof can be found in e.g. Dimarogonas and Johansson (2008); Deghat et al. (2016).

3. SELF-CONTAINED DISTANCE ERROR SYSTEMS AND THEIR EXPONENTIAL STABILITY

3.1 System equations for distance errors e

The convergence result in Lemma 1 focuses on the formation position system (4) (or (7)) and confirms the asymptotic convergence of the distance error vector e; however, it does not show whether the convergence of e to the origin is exponentially fast. Since in this paper our main focus is on the robustness analysis of the formation system under distance mismatches, it is natural to consider the dynamical system which describes the evolution of the distance errors e. For convenience of analysis we will adopt a compact form for the distance error equation. In the following we focus on the formation stabilization system (4) (without velocity consensus). The case for the formation flocking system (7) (with velocity consensus term) can be dealt with in a similar manner.

Bearing in mind that $\dot{e} = 2R\dot{p}$, we derive the following equations for $\ddot{e}$

$$\begin{aligned}
\ddot{e} &= 2R\ddot{p} + 2\dot{R}\dot{p} = -2\alpha Rv - 2RR^\top e + 2\dot{Z}^\top\bar{H}v\\
&= -2\alpha Rv - 2RR^\top e + 2\dot{Z}^\top\dot{z},
\end{aligned} \tag{9}$$

where $\dot{Z}^\top\dot{z} = [\|\dot{z}_1\|^2, \ldots, \|\dot{z}_k\|^2, \ldots, \|\dot{z}_m\|^2]^\top$. It has been proven in Mou et al. (2016); Sun et al. (2017) that when the formation shape is close to the desired one, the entries of the matrix $R(z)R^\top(z)$ are continuously differentiable functions of the distance error vector e. Hence we can write $M(e) := R(z)R^\top(z)$. Also note that $2\alpha Rv = \alpha\dot{e}$. By defining $\chi := \dot{e}$, one can rewrite (9) as

$$\begin{aligned}
\dot{e} &= \chi,\\
\dot{\chi} &= -\alpha\chi - 2M(e)e + 2\dot{Z}^\top\dot{z}.
\end{aligned} \tag{10}$$

The above equations will be particularly useful for conducting stability and robustness analysis in later sections. However, they do not constitute a set of self-contained equations due to the presence of the additional term $\|\dot{z}_k\|^2$. Note that from Lemma 1 and its proof using the standard argument of Barbalat's Lemma, one can conclude that each $||\dot{p}_i||$ is square integrable and tends to zero asymptotically, which implies that all $||\dot{z}_i||$ also have the same property. So the equation for $\ddot{e}_i$ is self-contained in the e_i except for the addition of a nonnegative term which converges to zero asymptotically, and which has a bounded integral. All these facts also justify the assumption in the following analysis that it is legitimate to linearize (10) around $(e = 0, \dot{e} = 0)$, which will be discussed in the next subsection through an augmented distance error system.

3.2 An augmented distance error system

Motivated in part by the above observation, we are now going to deal with the terms $||\dot{z}_i||^2$ of (10) in a different way, through arguing that they fall out when a certain set of self-contained equations is linearized. These self-contained equations contain more variables of course than just the e_i. This is a novel feature of the move from single integrator to double integrator agents.

We introduce a new variable

$$f_i = z_i \wedge \dot{z}_i, \tag{11}$$

and regard f_i as a vector describing a quantity relating to angular momentum associated with the i-th relative position.

Next, note that

$$||z_i||^2||\dot{z}_i||^2 = (z_i^\top \dot{z}_i)^2 + ||z_i \wedge \dot{z}_i||^2 = \frac{1}{2}\dot{e}_i^2 + |f_i|^2, \tag{12}$$

or

$$||\dot{z}_i||^2 = \frac{1}{||z_i||^2}\left(\frac{1}{2}\dot{e}_i^2 + |f_i|^2\right), \tag{13}$$

which indicates that the term $||\dot{z}_i||$, and in general all the entries in the stacked vector $\dot{Z}^\top \dot{z}$, can be considered as a function of $e, \dot{e}$ and f.

Following the above argument, we associate an angular momentum f_i with each edge and then analyze the equation of $\dot{f}_i$ for a general formation. Denote a vector $f = [f_1^\top, \dots, f_m^\top]^\top$. In the following, we define the $\wedge$ operation for two structured dm-dimensional vectors comprising a collection of m d-dimensional subvectors. In particular, we write

$$f = \begin{bmatrix} z_1 \wedge \dot{z}_1 \\ z_2 \wedge \dot{z}_2 \\ \vdots \\ z_m \wedge \dot{z}_m \end{bmatrix} =: z \wedge \dot{z}, \text{ and } \dot{f} = \begin{bmatrix} z_1 \wedge \ddot{z}_1 \\ z_2 \wedge \ddot{z}_2 \\ \vdots \\ z_m \wedge \ddot{z}_m \end{bmatrix} =: z \wedge \ddot{z}.$$

From the equation for the formation stabilization system (5), one can obtain

$$\begin{aligned} \dot{f} &= \dot{z} \wedge \dot{z} + z \wedge \ddot{z} = z \wedge \ddot{z} = z \wedge \left(-\alpha \dot{z} - \bar{H}R^\top(z)e\right) \\ &= -\alpha z \wedge \dot{z} - z \wedge (\bar{H}R^\top(z)e), \end{aligned} \tag{14}$$

where the entries of the vector function term $z \wedge (\bar{H}R^\top(z)e)$ involve linear combinations of terms like $e_j z_i \wedge z_j$. According to Lemma 1, the error vector e asymptotically converges to zero, which implies that in the vicinity of the limit $e = 0$, the term $z_i \wedge z_j$ is close to some bounded constant (actually its magnitude is twice the area of the triangle formed by the three agents associated with z_i and z_j). Following the same argument that the inner product term $z_i^\top z_j$ is a function of e, it is obvious that the wedge product $z_i \wedge z_j$ is also a function of e. The term $z \wedge (\bar{H}R^\top(z)e)$ is therefore of the form $G(z)e$ for some matrix G. From Lemma 1, the convergence of e to zero also implies that the product $e_j z_i \wedge z_j$ will also converge to zero when $t \to \infty$, and we conclude that in the limit there will also hold $f_i = 0$.

3.3 Local exponential convergence of distance error system via linearization analysis

Now we exhibit the linearized equations of the augmented system around the desired equilibrium point $\{(e, \chi, f) | e = 0, \chi = 0, f = 0\}$. The third observation that the term $\|\dot{z}_k\|^2$ is of second order in χ and f_i will be a key to recording a decoupled linearized system. The linearization equations for (10) and $\dot{f}_i$ around $(\bar{e} = 0, \bar{\chi} = 0, \bar{f} = 0)$ can be easily calculated as

$$\begin{bmatrix} \dot{\bar{e}} \\ \dot{\bar{\chi}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & I \\ -2M(0) & -\alpha I \end{bmatrix}}_{=: J_{(\bar{e},\bar{\chi})}} \begin{bmatrix} \bar{e} \\ \bar{\chi} \end{bmatrix}, \tag{15}$$

and

$$\dot{\bar{f}} = -\alpha \bar{f} - \bar{z} \wedge (\bar{H}R^\top(\bar{z})\bar{e}), \tag{16}$$

where in the linearized system (16) the entries of the vector function term $\bar{z} \wedge (\bar{H}R^\top(\bar{z})\bar{e})$ involve the linearized quantity $\bar{e}$ and $\bar{z}_i \wedge \bar{z}_j$ with $\bar{z}_i$ referring to the relative position from the resulted target formation with side length d_i. Thus, the term $\bar{z} \wedge (\bar{H}R^\top(\bar{z})\bar{e})$ is of the form $G(\bar{z})\bar{e}$ for some matrix G whose entries are functions of $\bar{z}$. It is clearly seen from the above linearized equations that the first equation (15) is *decoupled* from the second one (16). For the linearized error system (15), the matrix $-M(0)$ is negative definite (see e.g. Sun et al. (2016b) for a proof), which further implies that all the eigenvalues of the Jacobian matrix $J_{(\bar{e},\bar{\chi})}$ have negative real parts, i.e., the Jacobian is a Hurwitz matrix (see e.g. Sun et al. (2016b) for the proof). According to (Khalil, 2002, Theorem 4.13), this proves the local exponential convergence of the distance error system (10). The exponential convergence of $(e = 0, \dot{e} = 0)$ from (15), together with the structure of the linearized equation (16), also implies that $f = 0$ is locally exponentially convergent. We summarize all these results in the following theorem.

Theorem 1. The equilibrium state $(e = 0, \dot{e} = 0, f = 0)$ of the unperturbed augmented error system in (10) and (14) is locally exponentially stable.

For the formation flocking system, one can also derive the distance error system from (7) as follows

$$\ddot{e} = 2R\ddot{p} + 2\dot{R}\dot{p} = -2R\bar{H}^\top \dot{z} - 2RR^\top e + 2\dot{Z}^\top \dot{z}. \tag{17}$$

By augmenting additional variable $f = z \wedge \dot{z}$ with the system $\dot{f}$ being in a similar form as in (14), and following a similar analysis to the above argument (which is omitted here), one can also show the locally exponential convergence of the equilibrium state $(e = 0, \dot{e} = 0, f = 0)$ for the augmented distance error system derived from the formation flocking system (7). We summarize:

Theorem 2. The equilibrium state $(e = 0, \dot{e} = 0, f = 0)$ of the unperturbed augmented error system $\ddot{e}$ and $\dot{f}$ derived from the flocking formation system in (7) is locally exponentially stable.

The robustness property as a consequence of the exponential stability will be discussed in the next section.

4. ROBUSTNESS ISSUES AND MOTION PROPERTIES WITH DISTANCE MISMATCHES

4.1 Modified system equations with distance mismatches

Following the problem setting in Belabbas et al. (2012); Sun et al. (2014); Mou et al. (2016); Garcia de Marina et al. (2016), we now assume in this section that the perceived distances d_{ij} and d_{ji} for neighboring agents i and j, respectively, are not necessarily equal. Furthermore, the misbehavior actually stems from the mismatch (the difference, or discrepancy) between d_{ij} and d_{ji} rather than the assumption that both d_{ij} and d_{ji} are only approximately equal to $d_{k_{ij}}$. In other words, only the difference between mutual distances in each edge matters in the modelling of distance mismatch. Without loss of generality and to simplify the equations in the sequel, we will henceforth assume that d_{ij} exactly equals $d_{k_{ij}}$ for all adjacent vertex pairs (i, j) for which i is the head of edge k_{ij}. Next, we denote $\mu_{k_{ij}} = d_{ij}^2 - d_{ji}^2$ as the constant distance mismatch corresponding to edge k_{ij}; clearly, one has $d_{ij}^2 = d_{k_{ij}}^2, d_{ji}^2 = d_{k_{ij}}^2 - \mu_{k_{ij}}$. We also denote by $\mathcal{N}_i^+$ the set of all $j \in \mathcal{N}_i$ for which vertex i is the head of the oriented edge k_{ij}, and denote by $\mathcal{N}_i^-$ the complement of $\mathcal{N}_i^+$ in $\mathcal{N}_i$. Thus, the double-integrator formation stabilization system with distance mismatches should be modified as

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= -\alpha v_i - \sum_{j\in\mathcal{N}_i} e_{k_{ij}}(p_i - p_j) + \sum_{j\in\mathcal{N}_i^-} \mu_{k_{ij}}(p_i - p_j).\end{aligned}$$

Following again the same procedure from Belabbas et al. (2012); Mou et al. (2016); Garcia de Marina et al. (2016), one can further define J and $\bar{J}$ to be the matrices obtained from $-H$ and $-\bar{H}$ by replacing all -1 entries by zeros. With the definition of $\bar{J}$, we can define an $m \times 3n$ matrix $S(z)$ by $S(z) = Z^\top \bar{J}$, and the compact form of the formation stabilization system with distance mismatches is written as

$$\begin{aligned}\dot{p} &= v,\\ \dot{v} &= -\alpha v - R^\top(z)e(z) + S^\top(z)\mu,\end{aligned} \tag{18}$$

where $\mu = [\mu_1, \mu_2, \dots, \mu_m]^\top$ is a vector collecting all mismatched values for all the edges. Also, the formation distance error system should be modified as

$$\begin{aligned}\dot{e} &= \chi,\\ \dot{\chi} &= -\alpha\chi - 2M(e)e + 2\dot{Z}^\top \dot{z} + 2R(z)S^\top(z)\mu.\end{aligned} \tag{19}$$

Note that the mismatch term μ enters the distance error system in a linear sense, multiplied by the term $2R(z)S^\top(z)$. Furthermore, when the formation shape is close to the desired one, the entries of the matrix $R(z)S^\top(z)$ are continuously differentiable functions of the distance error vector e (see proofs in Mou et al. (2016); Sun et al. (2017)).

In a similar way, one can derive the mismatched version of the flocking formation system from (7) as

$$\begin{aligned}\dot{p} &= v,\\ \dot{v} &= -\bar{L}v - R^\top(z)e(z) + S^\top(z)\mu.\end{aligned} \tag{20}$$

and its associated distance error system in the following form

$$\ddot{e} = -2R\bar{H}^\top \dot{z} - 2RR^\top e + 2\dot{Z}^\top \dot{z} + S^\top(z)\mu. \tag{21}$$

4.2 Convergence of perturbed distance error system

From the exponential stability shown in Theorem 1 and Theorem 2, one can conclude the following convergence results for the distance error e. Note that the following lemmas hold for both formation stabilization system (19) and formation flocking system (21) with constant mismatches μ.

Lemma 2. Suppose initial conditions $(e(0), \dot{e}(0), f(0))$ are sufficiently close to the equilibrium $(e = 0, \dot{e} = 0, f = 0)$ for the error system (i.e. the system (19) or (21)). Then for sufficiently small and constant μ, the distance error $e(t) = e(\bar{H}p(t))$ converges exponentially fast to an equilibrium close to the origin.

It is well known that exponentially stable systems are robust to small perturbations, and the bounds of nonvanishing perturbations which do not destroy the exponential stability of the nominal systems can be given an inequality formula (see (Khalil, 2002, Lemma 9.3)). In our case the bounds depend on the exponential rate of the distance error system, the region of attraction, and the evolution of the rank of the rigidity matrix as well as the formation shape; unsurprisingly therefore, an explicit formula of the bound is hard to obtain. Returning now to the main argument, we denote the equilibrium of the error system under small and constant mismatches μ as $\bar{e}(\mu)$, or shortly as $\bar{e}$, which is a continuously differentiable function of μ. A further consequence of $e \to \bar{e}(\mu)$ is stated in the following lemma.

Lemma 3. Given the convergence of the distance error $e(t)$ to the equilibrium state $\bar{e}$, the inner product term, $z_k^\top z_k$ for all k and $z_i^\top z_j$ for $i \neq j$, will also converge to constants.

4.3 Rigid motions in double-integrator formation shape stabilization systems

The aim of this subsection is to show the formation behavior and motion property of the formation stabilization system modelled by double integrators in the presence of mismatched distances. From the expression of the mismatched version of formation stabilization system (18) and the convergence results shown in the above Subsection 4.2, one can prove the following facts.

Lemma 4. The norm of each agent's velocity, i.e. $\|\dot{p}_i\|$, is constant when $e(\bar{H}p(t)) = \bar{e}$ as described by (21). Furthermore, the norm of the formation centroid's velocity, i.e. $\|\dot{p}_c\| = \|\frac{1}{n}\sum_{i=1}^n \dot{p}_i\|$, is constant at the equilibrium motion when $e(\bar{H}p(t)) = \bar{e}$.

Lemma 4 implies that the speed of the agents and the norm of their accelerations are non-zero constants. Thus the velocity and acceleration vectors are perpendicular. By combining the above results, we derive the motion behavior of the formation stabilization caused by constant mismatches.

Theorem 3. In the presence of small and constant μ in the modified formation stabilization system (18), the formation shape converges exponentially fast to a rigid one, and $p(t)$ converges exponentially fast to a **circular orbit** (in the 2-D case) or a **helical orbit** (in the 3-D case) of the overall system (18) along which $e(\bar{H}p(t)) = \bar{e}$.

The proofs for Lemma 4 and Theorem 3 follow similarly the proof and analysis in Sun et al. (2017) and are omitted here due to space limit.

4.4 Rigid motions in double-integrator formation flocking systems

The aim of this subsection is to show the formation behavior and motion property of the double-integrator formation flocking system induced by mismatched distance. From the system equation of the mismatched version of formation flocking system (20) and the convergence results shown in Subsection 4.2, one can prove the following facts.

***Lemma** 5.* The norm of each agent's acceleration, i.e. $\|\ddot{p}_i\|$, is constant when $e(\bar{H}p(t)) = \bar{e}$. Furthermore, the norm of the formation centroid's acceleration, i.e. $\|\ddot{p}_c\|$, is constant at the equilibrium motion when $e(\bar{H}p(t)) = \bar{e}$.

By combining the result in the above lemma and the convergence results in Section 4.2, we conclude the motion behavior of the formation stabilization caused by constant mismatches in the following theorem.

***Theorem** 4.* In the presence of small and constant μ in the modified formation flocking system (20), the formation shape converges exponentially fast to a rigid one, and $\dot{p}(t)$ converges exponentially fast to a **circular orbit** (in the 2-D case) or a **helical orbit** (in the 3-D case) of the overall system (20) along which $e(\bar{H}p(t)) = \bar{e}$.

Note that as compared to Theorem 3 on the motion property for the formation stabilization system described by agents' positions $p(t)$, the above theorem on the formation flocking system establishes a similar result on agents' velocities $\dot{p}(t)$, while the steady-state trajectories $p(t)$ for all agents will be governed by the motion rule for $\dot{p}(t)$ along which $e(\bar{H}p(t)) = \bar{e}$. The proofs for Lemma 5 and Theorem 4 follow similarly the proof and analysis in Sun et al. (2017) and are omitted here due to space limit, which will be provided in the full version of this paper.

5. CONCLUSIONS

In this paper we have discussed the robustness issues of formation control systems modelled by double integrators with distance mismatches. Two kinds of double-integrator formation control systems are considered, one with velocity damping term (termed the formation stabilization system) and the other with velocity consensus term (termed formation flocking system). We discussed in detail the self-contained issue of the distance error system, by adding additional terms to obtain an augmented distance error system. Then the linearization analysis around the equilibrium of the origin reveals the exponential stability of the distance error system, which further implies the robustness property of double-integrator formation systems in the presence of small distance mismatches.

We have also discussed the effect of small constant mismatch term on the formation system, and show that (i) for double-integrator formation stabilization systems, the induced rigid motion is identical to that in single-integrator case (described by agents' positions); and (ii) for double-integrator formation flocking systems, the orbit of steady-state velocity displays the same type of trajectories as the motion property in single-integrator formation systems described by agents' position variables.

REFERENCES

Belabbas, A., Mou, S., Morse, A.S., and Anderson, B.D.O. (2012). Robustness issues with undirected formations. In *Proc. of the 51st Conference on Decision and Control*, 1445–1450. IEEE.

Deghat, M., Anderson, B.D.O., and Lin, Z. (2016). Combined flocking and distance-based shape control of multi-agent formations. *IEEE Transactions on Automatic Control*, 61(7), 1824–1837.

Dimarogonas, D.V. and Johansson, K.H. (2008). On the stability of distance-based formation control. In *Proc. of the 47th IEEE Conference on Decision and Control*, 1200–1205. IEEE.

Garcia de Marina, H., Cao, M., and Jayawardhana, B. (2015). Controlling rigid formations of mobile agents under inconsistent measurements. *IEEE Transactions on Robotics*, 31(1), 31–39.

Garcia de Marina, H., Jayawardhana, B., and Cao, M. (2016). Distributed rotational and translational maneuvering of rigid formations and their applications. *IEEE Transactions on Robotics*, 32(3), 684–696.

Garcia de Marina, H., Jayawardhana, B., and Cao, M. (2017). Taming mismatches in inter-agent distances for the formation-motion control of second-order agents. *IEEE Transactions on Automatic Control. Online version at arXiv preprint arXiv:1604.02943.*

Hendrickson, B. (1992). Conditions for unique graph realizations. *SIAM Journal on Computing*, 21(1), 65–84.

Khalil, H.K. (2002). *Nonlinear systems, the 3rd edition.* Prentice hall New Jersey.

Krick, L., Broucke, M.E., and Francis, B.A. (2009). Stabilisation of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82(3), 423–439.

Mou, S., Belabbas, M.A., Morse, A.S., Sun, Z., and Anderson, B.D.O. (2016). Undirected rigid formations are problematic. *IEEE Transactions on Automatic Control*, 61(10), 2821–2836.

Mou, S., Morse, A.S., and Anderson, B.D.O. (2014). Toward robust control of minimally rigid undirected formations. In *Proc. of the 53rd Conference on Decision and Control*, 643–647. IEEE.

Oh, K.K., Park, M.C., and Ahn, H.S. (2015). A survey of multi-agent formation control. *Automatica*, 53, 424–440.

Olfati-Saber, R. (2006). Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3), 401–420.

Olfati-Saber, R. and Murray, R.M. (2002). Graph rigidity and distributed formation stabilization of multi-vehicle systems. In *Proc. of the 41st IEEE Conference on Decision and Control*, volume 3, 2965–2971. IEEE.

Sun, Z., Mou, S., Anderson, B.D.O., and Morse, A.S. (2017). Rigid Motions of 3-D Undirected Formations with Mismatch between Desired Distances. *IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2017.2676680.*

Sun, Z., Anderson, B.D.O., Deghat, M., and Ahn, H.S. (2016a). Rigid formation control of double-integrator systems. In *International Journal of Control, in press, DOI: 10.1080/00207179.2016.1207100.*

Sun, Z., Mou, S., Anderson, B.D.O., and Cao, M. (2016b). Exponential stability for formation control systems with generalized controllers: a unified approach. *Systems and Control Letters*, 93, 50–57.

Sun, Z., Mou, S., Anderson, B.D.O., and Morse, A.S. (2014). Formation movements in minimally rigid formation control with mismatched mutual distances. In *Proc. of the 53rd Conference on Decision and Control*, 6161–6166. IEEE.

Tanner, H.G., Jadbabaie, A., and Pappas, G.J. (2007). Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control*, 52(5), 863–868.